Exact Duality and Bjorken Sum Rule in Heavy Quark Models à la Bakamjian-Thomas

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Abstract

The heavy mass limit of quark models based on the Bakamjian-Thomas construction reveals remarkable features. In addition to previously demonstrated properties of covariance and Isgur-Wise scaling, exact duality, leading to the Bjorken-Isgur-Wise sum rule, is proven, for the first time to our knowledge in relativistic quark models. Inelastic as well as elastic contributions to the sum rule are then discussed in terms of ground state averages of a few number of operators corresponding to the nonrelativistic dipole operator and various relativistic corrections.

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1 Introduction

In [1] we have proposed a quark model of current matrix-elements in the heavy quark limit which has been shown to present two important features : it is covariant and it presents the full scaling properties of heavy quark symmetry (HQS), demonstrated by Isgur and Wise for QCD [2]. The model is based on an old formulation of relativistic multiparticle states, i.e. a representation of the full Poincaré group, proposed by Bakamjian and Thomas [3] and reexpressed by Osborn [4]. We have given a simpler and concise description of Poincaré generators and wave functions in momentum space [1]. Since some time this formulation, or variants of it, have been used by a large variety of people to formulate quark models of form factors, especially on the null-plane [5]. One must stress that by boosting the states to arbitrarily large momenta, one recovers the models directly formulated on the null-plane. In particular, one gets null-plane wave functions with the correct kinematical limits, unlike other approaches.

It must be emphasized however that in this formulation, for finite quark masses, the currents based on the free quark current operators are not covariant. What we have found is that their *heavy quark mass limit are covariant*. One important consequence of this covariance property of the limit is that all the models we have referred to will have essentially the same limit, whichever the frame in which they have been formulated when $m_Q < \infty$. Then, the only remaining source of variation consists in the choice of the mass operator or of the set of wave functions at rest, and everything we are to say, being completely independent of such choices, should hold for the $m_Q \to \infty$ limit of such models.

The implementation of covariance and heavy quark symmetry obtained in this approach must be fully appreciated. Indeed they are not built in, neither enforced by hand. It represents an important progress with respect to older popular models of heavy-to-heavy semi-leptonic form factors which, on the one hand, make implicit or explicit reference to particular frames to calculate form factors, and, which, on the other hand, finally renounce to predict form factors except for a privileged q^2 value, and inspire for instance from VMD to extrapolate form factors. Then either heavy quark symmetry is not satisfied or it is enforced by hand. In the present approach, on the contrary, everything results from a systematic calculation from the eigenstates of a mass operator, and a current operator. It must be also underlined that the approach, while leading to such nontrivial properties, does not spoil the simplicity which makes quark models so attractive. It preserves principles of quark models like a fixed number of constituents, a three-dimensional description through ordinary wave-functions, a free quark current operator . On the other hand, it is also a progress with respect to our own older models [6] which, while presenting heavy quark symmetry, were only approximately covariant (i.e. the Isgur-Wise scaling function ξ was depending on the chosen frame).

In the present paper, we pursue the investigation of the model presented in [1] as regards general properties, not dependent on particular choices of interaction or wave functions at rest : the mass operator will remain totally arbitrary, except of course for rotation and parity invariance. As another important advantage of the new approach, we want now to show that it may help a lot in understanding more physically, in the context of bound state physics, *sum rules* already formulated in a field-theoretical context. Let us stress that exact saturation of sum rules is a rather specific feature of this type of relativistic models. Sum rules strongly rely on completeness relations. But what is precisely needed is completeness relations for wave functions of states in *motion*. Such

relations are automatically provided by the construction of a unitary representation of the Poincaré group. On the other hand, many phenomenological models use wave functions in motion which are not shown to be orthonormal, e.g. to be eigenstates of an Hamiltonian or even, as recalled above, use form factors not calculated through wave functions except for a normalisation at some q^2. Then, they have no reason to satisfy sum rules.

In addition to demonstrating that Bjorken-Isgur-Wise (BIW) sum rule is exactly valid in our approach, we shall also decompose the contributions of the various states to the "physical side" of the sum rule into different parts with clear physical meaning, clarifiying in particular the nature of relativistic corrections and the origin of the various bounds on ρ^2.

2 Direct demonstration of duality and saturation of the Bjorken-Isgur-Wise sum rule

The sum rule under consideration writes, for mesons, in the transparent formulation of Isgur and Wise [8]

$$\rho^2 - \sum_k \left(\left| \tau^k_{1/2}(1) \right|^2 + 2 \left| \tau^k_{3/2}(1) \right|^2 \right) = \frac{1}{4} \tag{1}$$

where ρ^2 is the slope of the elastic ground state Isgur-Wise scaling function, while the $\tau_{1/2}$, $\tau_{3/2}$ are the scaling functions of the transition between the ground state and the P wave ($L = 1$) states with possible radial excitation, $k = 1$ for the lowest P state, $k = 2,$. for its radial excitations. In the following, the τ_i's are always considered at the zero-recoil point $w = 1$. Therefore, to reduce notation, we *denote the corresponding values by* $\tau_{1/2}$, $\tau_{3/2}$.

To demonstrate the sum rule, we shall not use the original Bjorken method starting from commutators of field theory [7], since our model does not contain a priori such commutation relations. Rather, we shall calculate directly the sum of squares of transition matrix elements to any final state and find that it has a very simple value, independent of the dynamics. In the present approach, states have a fixed number of constituents N (no pair creation). Moreover, for a confining interaction, they will be stable bound states ; this implies that saturation of the sum rule holds within the resonance-dominance and narrow-resonance approximations.

For the particular purpose of this article, it is found easier to treat on equal footing the various states with different spins and which may be mesons or baryons or states with any number of quarks ; we then abandon the manifestly covariant formalism with Dirac spinors, and return to the initial bidimensional-spinor formalism. In this formalism, current density matrix-elements write, without assuming m_1, $m'_1 \to \infty$:

$$< \vec{P}'|j^\mu|\vec{P} >= \int \prod_i d^3p_i \sum_{s_i s_1 s'_1} \psi^*_{s'_1 s_i}(\vec{P}' - \Sigma\vec{p}_i, \{\vec{p}_i\})$$

$$< \vec{p'}_1 s'_1|j^\mu|\vec{p}_1 s_1 > \psi_{s_1 s_i}(\vec{P} - \Sigma\vec{p}_i, \{\vec{p}_i\}). \tag{2}$$

Index 1 denotes the active quark, while spectator quarks, from 2 to N, are denoted by the generic index i. We disregard color and flavor. The ψ's are what we call the "relative"

wave functions. They are obtained by dropping $\delta(\Sigma\vec{p}_i - \vec{P})$ in the total wave function with total momentum $\vec{P}$ (we normalize any state to $\delta(\vec{P}' - \vec{P})$ and 1 for any discrete index).

Part of our demonstration will be worked out without passing to the heavy quark limit. Let us recall that at this stage the model is not covariant, but we do not need the covariance property. For the first and most general demonstration of the sum rule, we do not need either to make explicit the construction of the ψ's from the internal wave functions at rest. This construction shall be explained in detail in section 3, where one really needs it. We need only to know that the ψ's may be chosen to form a complete orthonormal basis of the subspace with $\vec{P}$ fixed. Let us label it by a generic label n in addition to $\vec{P}$, we mean that :

$$\sum_{s_1 s_i} \int \prod_i d^3p_i \; \psi^*_{n' s_1 s_i}(\vec{P} - \Sigma\vec{p}_i, \{\vec{p}_i\}) \psi_{n s_1 s_i}(\vec{P} - \Sigma\vec{p}_i, \{\vec{p}_i\}) = \delta_{nn'} \quad . \tag{3}$$

Then we have the closure relation :

$$\sum_n \psi_{n s_1 s_i}(\vec{P} - \sum_i \vec{p}_i, \{\vec{p}_i\}) \psi^*_{n s'_1 s'_i}(\vec{P} - \Sigma\vec{p'}_i, \{\vec{p'}_i\}) = \delta_{s_1 s'_1} \prod_i \delta_{s_i s'_i} \prod_i \delta(\vec{p}_i - \vec{p'}_i) \quad . \tag{4}$$

It is of course convenient to choose the basis among the eigenstates of energy $\sqrt{M_{op}^2 + \vec{P}^2}$, and to choose for the label n the labelling of the corresponding internal wave functions at rest $\varphi_n(\vec{k}_1, \{k_i\})$, which will include, in addition to various internal spin and angular momentum labels, excitation numbers.

Let us now consider the transition matrix-elements from one fixed state denoted as $n = 0$, $\vec{P}$, to all possible states, i.e. with all possible n, with another common momentum $\vec{P}'$, induced by j^μ. We choose for j^μ the elastic vector current, although we could choose any other current and even any Dirac matrix $\mathcal{O}$, with $m_1 \neq m'_1$ - indeed, it is known that whatever the choice, we would obtain the same Bjorken sum rule because of HQS. Our choice is made for transparency and definiteness only.

With the closure relation eq.(4) at hand, one can perform the following sum :

$$h^{\mu\nu}(\vec{P}, \vec{P}') = \sum_n < n, \vec{P}'|j^\mu|0, \vec{P} > \times (< n, \vec{P}'|j^\nu|0, \vec{P} >)^* \tag{5}$$

which we will call the "hadronic tensor" in analogy with inclusive leptoproduction or semi-leptonic decays. Let us stress that this sum should not be covariant in principle, even if the theory were covariant, because each intermediate state has a different energy at fixed $\vec{P}'$. We find trivially from eqs. (2 and 4) :

$$h^{\mu\nu}(\vec{P}, \vec{P}') = \sum_{s'_1 s_1 \hat{s}_1 s_i} \int \prod_{i=2}^N d^3p_i \; \psi^*_{0 \hat{s}_1 s_i}(\vec{P}' - \sum_i \vec{p}_i, \{\vec{p}_i\}) \psi_{0 s_1 s_i}(\vec{P} - \sum_i \vec{p}_i, \{\vec{p}_i\})$$

$$(\bar{u}_{s'_1}(\vec{P}' - \Sigma\vec{p}_i)\gamma^\mu u_{s_1}(\vec{P} - \Sigma\vec{p}_i)) \times (\bar{u}_{s'_1}(\vec{P}' - \Sigma\vec{p}_i)\gamma^\nu u_{\hat{s}_1}(\vec{P} - \Sigma\vec{p}_i))^* \quad . \tag{6}$$

The important step already realized is to have reexpressed the hadronic tensor in terms of the initial state ($n = 0$) wave function only. In fact, the right-hand side of the equation is the average of a rather simple operator. Introducing the Fourier transform of the current density :

$$\tilde{j}(\vec{q}) \equiv \int d^3x\ e^{-i\vec{q}\cdot\vec{x}}\ j(\vec{x}) \tag{7}$$

one can write :

$$h^{\mu\nu}(\vec{P},\vec{P}') =< 0;\vec{P}\left|\tilde{j}^{\nu}(\vec{q})\ j^{\mu}(0)\right|0;\vec{P}> \tag{8}$$

with $\vec{q} \equiv \vec{P}' - \vec{P}$.

Up to now all the written relations were valid in full generality. Let us now take advantage of the heavy quark mass limit $m_1 \to \infty$. The important point is that in this limit, the velocity of the active quark can be considered equal to the velocity of the corresponding hadron, $\vec{v}$ or $\vec{v}'$. Then we can factor out of the integral the spinor factors, which now depend only on $\vec{v}$ and $\vec{v}'$:

$$h^{\mu\nu}(\vec{v},\vec{v}') = \sum_{s_1 s_1' \widehat{s}_1} \left[\bar{u}_{s_1'}(\vec{v}')\gamma^{\mu}u_{s_1}(\vec{v})\right]\left[\bar{u}_{s_1'}(\vec{v}')\gamma^{\nu}u_{\widehat{s}_1}(\vec{v})\right]^*$$

$$\sum_{s_i}\int\prod_{i=2}^{N} d^3p_i\ \psi^*_{0\widehat{s}_1 s_i}(\vec{P}-\Sigma\vec{p}_i,\{\vec{p}_i\})\psi_{0s_1s_i}(\vec{P}-\Sigma\vec{p}_i,\{\vec{p}_i\}) \quad . \tag{9}$$

A further simplification can be obtained if one averages as over the polarisation of the initial state 0 ; let us call it λ. Then it is easy to convince oneself that :

$$\int\prod_{i=2}^{N} d^3p_i \frac{1}{2J_0+1}\sum_{s_i,\lambda}\psi^{\lambda *}_{0\widehat{s}_1 s_i}\ \psi^{\lambda}_{0s_1s_i} \propto \delta_{s_1\widehat{s}_1} \quad . \tag{10}$$

First, this is easily seen to follow from rotation invariance if $\vec{P}=0$. Then, one can extend the result to arbitrary $\vec{P}$ using the following property of the model : in passing to wave functions in motion, the spectator spins s_1 are simply rotated by Wigner rotations, which are cancelled by contraction, while for the active quark spin s_1,$\widehat{s}_1$, the Wigner rotation reduces to unity in the $m_1 \to \infty$ limit. Then

$$\int\prod_{i=2}^{N} d^3p_i \frac{1}{2J_0+1}\sum_{s_i\lambda}\psi^{\lambda *}_{0\widehat{s}_1 s_i}\ \psi^{\lambda}_{0s_1s_i} = \int\prod_{i=2}^{N} d^3p_i \frac{1}{2}\frac{1}{2J_0+1}\left(\sum_{s_1s_i\lambda}\psi^{\lambda *}_{0s_1 s_i}\ \psi^{\lambda}_{0s_1s_i}\right)\delta_{s_1\widehat{s}_1} \quad . \tag{11}$$

Whence our central relation :

$$\bar{h}_{\mu\nu}(\vec{v},\vec{v}') \equiv \frac{1}{2J_0+1}\sum_{\lambda} h^{\lambda}_{\mu\nu}(\vec{v},\vec{v}')$$

$$= \frac{1}{2}\sum_{s_1,s_1'}\left[\bar{u}_{s_1'}(\vec{v}')\gamma^{\mu}u_{s_1}(\vec{v})\right]\left[\bar{u}_{s_1'}(\vec{v}')\gamma^{\mu}u_{s_1}(\vec{v})\right]^* \equiv \bar{h}^{free\ quark}_{\mu\nu} \quad . \tag{12}$$

We have *exact duality* for the hadronic tensor, that is the general Bjorken sum rule in the form of Isgur and Wise. Let us recall that it holds with any Dirac space matrices instead of γ^{μ}, γ^{ν}.

Though it may appear trivial, it is worth stressing once more that such an exact duality can hold only because *current matrix elements* are actually calculated as matrix elements of a given operator between wave functions in motion that satisfy closure. If, on

the contrary, form factors are constructed according to phenomenological recipes, although one may have at start orthonormal wave functions at rest, one cannot get duality for arbitrary q^2 or $w = v \cdot v'$. Two properties of this heavy mass duality must also be underlined. First, the limit of $h_{\mu\nu}$ is manifestly covariant, in contrast to $h_{\mu\nu}$ at finite m_Q. Second, this limit is the sum of the heavy mass limit of each product of two matrix elements. This nonobvious exchange of limit and sum can be shown to hold within the model, using closure for the wave functions in the heavy mass limit. That entails that the sum of limits is convergent, which should not be necessarily true in general in field theory.

From this duality relation, one deduces easily the sum rule (1) by reexpressing the current matrix elements in the definition of $h^{\mu\nu}(\vec{P}, \vec{P}')$ (eq.(5)) in terms of the standard invariant scaling functions, [8] . The deduction will be made by expanding h_{00} to second order around $\vec{v}, \vec{v}' = 0$; for simplicity we choose $\vec{v}$ and $\vec{v}'$ collinear. Moreover, we take as initial state a 0^- meson. One finds, taking into account that our own normalisation of states is $\delta^3(\vec{P} - \vec{P}')$, different from the one of Isgur and Wise :

$$< 0^-; \vec{v}'|j^0|0^-; \vec{v} >= \frac{1}{2}\frac{v_0 + v_0'}{\sqrt{v_0 v_0'}}\xi \approx \xi \approx 1 - \frac{1}{2}\rho^2(\vec{v} - \vec{v}')^2 \quad . \tag{13}$$

At this point, we make use of the fact, demonstrated in the preceding letter, that the model indeed satisfies Isgur-Wise scaling, thanks in particular to the factorisation of the wave function at rest into spin and rotationally invariant space wave functions. This result extends to radial excitations, by dropping the 1 in eq. (13), because of orthogonality of wave functions at rest. On the other hand, we have not yet presented the demonstration of covariance and scaling for the general case, which we postpone to another paper. We assume for the moment this general result. Manifestly covariant and scaling expressions will be presented in a forthcoming paper for the transitions to $L = 1$ (P wave states) [10]. 0^- radial excitations yield $\mathcal{O}((\vec{v}' - \vec{v})^2)$ amplitudes, therefore a negligible $\mathcal{O}((\vec{v}' - \vec{v})^4)$ contribution to h_{00}. The other possible final states through second order in $\vec{v}, \vec{v}'$ in the sum rule are 1^- and the $L = 1$ 0^+, 1^+, 2^+ states, because j^0 is scalar under rotations. $0^- \to 0^+$ vanishes identically for a vector current. $0^- \to 1^-$ and 2^+ vanish since for j^0 they have the form $\vec{v} \times \vec{v}'$ from covariance. Therefore, in addition to the elastic $0^- \to 0^-$ contribution, we have just the $0^- \to 1^+$ one, to express in terms of the scaling functions at zero recoil τ_i. This is readily done :

$$< 1/2, \vec{\varepsilon}_\lambda^{\,*}|j^0|0^- >\approx -\tau_{1/2}\; \vec{\varepsilon}_\lambda^{\,*} \cdot (\vec{v}' - \vec{v}) \quad (\lambda = 0) \tag{14}$$

$$< 3/2, \vec{\varepsilon}_\lambda^{\,*}|j^0|0^- >\approx -\sqrt{2}\,\tau_{3/2}\, \vec{\varepsilon}_\lambda^{\,*} \cdot (\vec{v}' - \vec{v}) \quad (\lambda = 0) \quad . \tag{15}$$

We have made an expansion only at the lowest, first order in $\vec{v}, \vec{v}'$, since these expressions appear in squares (therefore we have made $v_0, v_0' \approx 1$). Then :

$$\sum_\lambda \left|< 1/2, \lambda|j^0|0^- >\right|^2 + \sum_\lambda \left|< 3/2, \lambda|j^0|0^- >\right|^2 = \left(|\tau_{1/2}|^2 + 2|\tau_{3/2}|^2\right)(\vec{v} - \vec{v}')^2 \quad . \tag{16}$$

Finally, summing on the P-wave states, including excitations :

$$h_{00}(\vec{v}, \vec{v}') \approx 1 - \rho^2(\vec{v} - \vec{v}')^2 + \sum_k \left(|\tau_{1/2}^k|^2 + 2|\tau_{3/2}^k|^2\right)(\vec{v} - \vec{v}')^2 \quad . \tag{17}$$

On the other hand, the right-hand side of eq. (12) is :

$$\bar{h}_{00}^{free\ quark} \approx 1 - \frac{1}{4}(\vec{v} - \vec{v}')^2 \tag{18}$$

whence the desired sum rule eq. (1).

For baryons, the difference will be that the scaling functions for the elastic transition are defined through the coefficient of γ^μ instead of $v^\mu + v'^\mu$, whence :

$$< 1/2^+|j^0|1/2^+ >\sim 1 - \frac{1}{2}\rho_B^2(\vec{v} - \vec{v}')^2 - \frac{1}{8}(\vec{v} - \vec{v}')^2 \tag{19}$$

because :

$$\bar{u}_{B'}\gamma^0 u_B \sim 1 - \frac{1}{8}(\vec{v} - \vec{v}')^2 \tag{20}$$

with our normalisations ($u_B^+ u_B = 1$). In this case, the sum rule does not contain the 1/4 term, but this has no deep physical meaning :

$$\rho_B^2 = \sum \ \text{inelastic contributions} \quad . \tag{21}$$

One must note that one is dealing with the usual Bjorken sum rules, corresponding to the first term in the current commutators considered by Bjorken ("direct" contribution) . It has been shown by Bjorken that the direct and z-graph contributions satisfy two separate, independent sum rules [7]. In our model as it stands, the z-graph contribution is absent due to the absence of pair creation or annihilation, therefore the second sum rules are not present. But a slight modification of the model, including pair creation or annihilation by the current and extending correspondingly the space of states, would probably allow to get the commutation relations and to satisfy the second type of sum rules corresponding to the z-graph contribution.

3 Analysis in terms of internal wave function matrix elements

Now we pass to a more detailed calculation of the various contributions to the left-hand side of the sum rule, as expressed in terms of internal wave functions at rest φ, eigenstates of the mass operator. That is, we display the various effects due to the hadron center-of-mass motion as treated relativistically. Then, we get more physical insight. We use the expression (12) of the preceding paper [1], obtained by explicitation of the relative wave function in terms of φ's (this is before passing to the $m_Q \to \infty$ limit)

$$< \vec{P}\ '|j^\mu|\vec{P} >= \int \prod_{i=2}^{N} d^3p_i \times \sqrt{\frac{\Sigma p'^0_j \Sigma p^0_j}{M'_0 M_0}} \prod_{j=1}^{N} \frac{\sqrt{k'^0_j k^0_j}}{\sqrt{p'^0_j p^0_j}} \times$$

$$\sum_{s_j}\sum_{s'_j} \varphi'_{s'_j}(\{\vec{k}'_i\}) \left[D(R'^{-1}_1)_{s'_1 s'''_1} \bar{u}_{s'''_1}\gamma^\mu u_{s''_1} D(R_1)_{s''_1 s_1}\right] \prod_{i=2}^{N} D(R'^{-1}_i R_i)_{s'_i s_i}\varphi_{s_j}(\{\vec{k}_i\}) \quad . \tag{22}$$

The $\vec{k}_i$ and $\vec{k}_i'$'s are the internal momenta corresponding respectively to the initial and final hadrons. They are complicated functions of the spectator momenta $\vec{p}_i$, obtained through the equations (4) of the preceding paper [1]. $p_{1,i}^0$ and k_i^0 denote simply $\sqrt{m_{1,i}^2+\vec{p}_{1,i}^{\,2}}$ and $\sqrt{m_i^2+\vec{k}_i^2}$ and $M_0 \equiv \sqrt{\Sigma k_i^2}$, j runs from 1 to N. R_j are Wigner rotations defined in the same paper. Let us recall how these Wigner rotations are practically related to boosts. Let Λ be a Lorentz boost :

$$\sqrt{p^0}\ \Lambda\ u_s(p) = \sqrt{(\Lambda p)^0}\ D_{ss'}(R)\ u_{s'}(\Lambda p) \tag{23}$$

with the rotation R defined by :

$$R = B_{\Lambda p}^{-1} \Lambda B_p \tag{24}$$

B_a is the boost which applies $(\sqrt{a^2}, \vec{0})$ on the four vector a_μ. The root factors in the formula are needed because of our normalization of states. From this formula, one can deduce easily $D_{ss'}(R)$ knowing Λ and p.

Let us identify the origin of the various factors involved in eq. (22), in addition to the wave functions φ's and to the free quark current density. In front of φ', there is the product of square roots of the Jacobians corresponding to the change of variables $\vec{p}_1, \{\vec{p}_i\} \to \{\vec{k}_i\}, \vec{P}$ and $\vec{p}\,'_1, \{\vec{p}_i\} \to \{\vec{k}_i\}, \vec{P}'$; the presence of these factors ensures unitarity of the corresponding functional transformations. Between φ' and φ, the Wigner rotations of quark spins ensure the passage from the ordinary one-particle spins to the internal spins ; note that the D matrices are unitary by themselves.

In the limit $m_Q \equiv m_1$ or $m_1' \to \infty$, the expression simplifies considerably. For convenience, we choose $\vec{v}$ and $\vec{v}'$ along the same axis $0z$ and count them by the algebraic numbers v_z and v_z'. We maintain $\vec{v} \neq 0$ to check the requirements of covariance on matrix elements, as imposed by the expressions (13), (14), (15). First, the relation between the k_i's and p_i's becomes :

$$k_i^0 = v^0 p_i^0 - v_z p_{iz}, \qquad k_{iz} = v^0 p_{iz} - v_z p_i^0, \qquad \vec{k}_{iT} = \vec{p}_{iT}, \tag{25}$$

where T denote the component perpendicular to $0z$. One has an analogous relation for the $\vec{k}_i'$'s with v' instead of v. Then, the Jacobian factors take the very simple form :

$$\prod_i \frac{\sqrt{v^0 p_i^0 - v_z p_{iz}}\sqrt{v'^0 p_i^0 - v_z' p_{iz}}}{p_i^0} \quad . \tag{26}$$

The Wigner rotations of the active quark tend to unity because its momentum becomes parallel to the direction of the boost, which becomes $\vec{v}$ or $\vec{v}'$. Finally, the active quark current density tends to :

$$\bar{u}_{s_1'}\ \gamma^0\ u_{s_1} \approx \left[1 - \frac{1}{8}(\vec{v}-\vec{v}')^2\right] \delta_{s_1' s_1} \tag{27}$$

and can be once more factored out of the integral. Whence the final expression in the limit $m_Q \to \infty$

$$< \vec{P}'|j^\mu|\vec{P} >= \sum_{s_1' s_1} \bar{u}_{s_1'}\gamma^\mu u_{s_1} \int \prod_{i=2}^{N} d^3p_i \prod_{i=2}^{N} \frac{\sqrt{(p_i \cdot v)(p_i \cdot v')}}{p_i^0}$$

$$\sum_{s_i s_i'} \varphi_{s_1' s_i'}(\{\vec{k}_i'\}) \prod_{i=2}^{N} D(R_i'^{-1} R_i)_{s_i' s_i} \ \varphi_{s_1 s_i}(\{\vec{k}_i\}) \quad . \tag{28}$$

Let us now expand the various factors in this expression around $\vec{v}$, $\vec{v}' = 0$, at fixed $\vec{p}_i$, $\vec{k}_i$ being a function of $\vec{p}_i$ and $\vec{v}$. We note that at $\vec{v} = \vec{v}' = 0$, one gets simply the scalar product of the internal wave functions at rest φ, φ'. Therefore it is 1 for the elastic transition $0 \to 0$ and 0 for inelastic transitions.

Then, to calculate the τ_i 's at zero recoil, we have once more to calculate inelastic amplitudes only through lowest, i.e. first order $\mathcal{O}(\vec{v})$, or $\mathcal{O}(\vec{v}')$. Let us first calculate these inelastic contributions. We have the following lowest order expansion of the integral :

$$< n|j^0|0 >= \int \prod_i d^3p_i \sum_i \varphi'^*(\vec{p}_i) \left[(v_z' - v_z) \left(p_i^0 \frac{\partial}{\partial p_{iz}} + \frac{\partial}{\partial p_{iz}} p_i^0 \right) \Big/ 2 \right.$$
$$\left. + \frac{i}{2}(v_z' - v_z) \frac{(\vec{\sigma} \times \vec{p}_{iT})_z}{p_i^0 + m_i} \right] \varphi(\vec{p}_i) \tag{29}$$

where scalar product on spin space is implied. The first term comes from the combination of the variation of the Jacobian factors, and the variation of the argument k of the wave function, the second one from Wigner rotations. Each has separately a factor $v_z' - v_z$, as required by covariance. Indeed, the covariance requires a factor $(v_z' - v_z)$ for the sum ; now, if we had dropped the spin and considered scalar quarks, the first term only would remain ; covariance then requires a factor $v_z' - v_z$ on this factor separately. Note that in this approximation the active quark density $\bar{u}_{s_1} \gamma^0 u_{s_1} \approx 1$ does not give a contribution. We can replace $i\ \partial/\partial p_{iz}$ by the more suggestive notation z_i ; it is indeed the space coordinate operator. In addition, we henceforth denote matrix elements between φ_n, $\varphi_{n'}$ as $(n'||n)$. Then :

$$< n|j^0|0 >\approx (v_z' - v_z) \sum_i \left(n \left| - \frac{p_i^0 i z_i + i z_i p_i^0}{2} + \frac{i}{2} \frac{(\vec{\sigma}_i \times \vec{p}_{iT})_z}{p_i^0 + m} \right| 0 \right) \quad . \tag{30}$$

If we particularize to mesons, $i = 2$ only ; we can drop the index i and write :

$$< P_{wave}|j^0|S_{wave} >\approx (v_z' - v_z) \left(n \left| - \frac{p^0 i z + i z p^0}{2} + \frac{i}{2} \frac{(\vec{\sigma} \times \vec{p}_T)_z}{p^0 + m} \right| 0 \right) \quad . \tag{31}$$

We now particularize to a 0^- initial state, and to $n = 1^+$. Then, one finds from the identification with eq. (16):

$$\sum_k \left(\left|\tau_{1/2}^k\right|^2 + 2\left|\tau_{3/2}^k\right|^2 \right) = \sum_{1/2,3/2} \sum_k \left| \left(1^+, k \left| - \frac{p^0 i z + i z p^0}{2} + \frac{i}{2} \frac{(\vec{\sigma} \times \vec{p}_T)_z}{p^0 + m} \right| 0^- \right) \right|^2 \quad . \tag{32}$$

It is clear that the operator on the right-hand side leads only to transitions to 1^+ (it has $L = 1$). Therefore we can as well replace the summation on states by a summation on the whole Hilbert space, and using closure of the φ's, we end with :

$$\sum_k \left(\left|\tau_{1/2}^k\right|^2 + 2\left|\tau_{3/2}^k\right| \right)^2 = \left(0^- \left| \left(\frac{p^0 z + z p^0}{2} \right)^2 + \frac{1}{4} \left[\frac{(\vec{\sigma} \times \vec{p}_T)_z}{p^0 + m} \right]^2 \right| 0^- \right) \quad . \tag{33}$$

We have taken into account that $\vec{\sigma}$ average on $S=0$ states is 0, so that the two contributions add in squares. Further simplification is obtained by using rotation invariance in the Wigner rotation contribution :

$$\sum_k \left(\left| \tau_{1/2}^k \right|^2 + 2 \left| \tau_{3/2}^k \right|^2 \right) = \left(0^- \left| \left(\frac{p^0 z + z p^0}{2} \right)^2 + \frac{1}{6} \frac{\vec{p}^2}{(p^0+m)^2} \right| 0^- \right) \tag{34}$$

which is an average on space wave functions only.

As to ρ^2, we could simply refer to the preceding paper, eq. (29) of the published version[1]. However, it is more instructive physically to recalculate ρ^2 along the same lines as we have just done for the τ's, i.e. directly in the formalism with bidimensional spin and Wigner rotations. We have just to push the expansion in $\vec{v}$ and $\vec{v}'$ of the matrix element through second order. It happens that the result can be decomposed in a manner similar to the τ's. One has three types of effects which make the matrix element depart from its zero recoil value, 1 :

i) one from the Jacobians and the spatial wave function arguments variations

ii) one from the Wigner rotations

iii) one from the current the active quark, which was not present for P waves.

noi Covariance requires the whole result to be of the form $1-(\rho^2/2)(\vec{v}'-\vec{v})^2$. But in fact each effect gives a separate contribution, which is $\propto (\vec{v}'-\vec{v})^2$. Indeed for iii) it is obvious, see eq. (37) below (it is the requirement of covariance for a one quark state). Then the effects i) and ii) must also combine to give a contribution of this form. Now, there are no crossed terms between them ; these should correspond to the product of factors of first order in the velocity : first order terms from the Wigner rotation effect ii) contain a spin operator $\vec{\sigma}$, which averages to 0 on a 0^- state ; then, these first order terms cannot combine with one from i) (the latter effect does not generate spin operators). i) and ii) give therefore non-interfering additive contributions. Moreover each one must have the form $(\vec{v}'-\vec{v})^2$. For the first effect i), it is seen by dropping spin and considering scalar quarks : then i) would give the only contribution and therefore it must be $\propto (\vec{v}'-\vec{v})^2$ separately by covariance. It results that the same form must holds for ii). The contribution i) to ρ^2 is found to be, by a somewhat lengthy calculation :

$$\rho^2_{space} = \left(0 \left| \left(\frac{p^0 z + z p^0}{2} \right)^2 \right| 0 \right) \quad . \tag{35}$$

The contribution ii) is

$$\rho^2_{wigner} = \left(0 \left| \frac{1}{6} \frac{\vec{p}^{\,2}}{(p^0+m)^2} \right| 0 \right) \quad . \tag{36}$$

Finally, from the active quark current :

$$< p_1' s_1' | j^0 | p_1 s_1 >= 1 - \frac{1}{8} (\vec{v}'-\vec{v})^2 \tag{37}$$

one has :

$$\rho^2_{dirac} = \frac{1}{4} \tag{38}$$

whence :

$$\rho^2=\rho^2_{space}+\rho^2_{wigner}+\rho^2_{dirac}=\left(0\left|\left(\frac{p_0z+zp_0}{2}\right)^2+\frac{1}{6}\frac{\vec{p}^{\,2}}{(p_0+m)^2}\right|0\right)+\frac{1}{4} \tag{39}$$

BIW sum rule is obviously satisfied, considering eq. (34) .

It is seen that "spatial" contributions to ρ^2 and τ^2's, i.e. the one obtained by dropping quark spin, cancel each other as it should since, for *scalar* quarks, the sum rule should write :

$$\rho^2-\sum_k\left|\tau^k\right|^2=0 \tag{40}$$

since in this fictitious case $h_{00}^{free\ quark}\approx 1$ instead of $1-1/4(\vec{v}-\vec{v}')^2$. Wigner rotation contributions cancel also each other, which is seen to be due to the unitarity of Wigner rotations.

Only the 1/4 coming for the active quark current has no counterpart in the τ^2's. One notes once more that this 1/4 is present in the j^0 matrix element of any elastic transition. It *will be present also for a free quark*. Let us also recall that the absence of 1/4 in baryon sum rules has nothing physical : it is due to the definition of invariant form factors, which are differently related to j^0 matrix elements. In terms of the latter, the only difference between mesons and baryons comes from the fact that i sums run over two spectator quarks instead of one ; but this refers to what we can term the *compositeness contribution* $\rho^2_{space}+\rho^2_{wigner}$.

In the following two subsections, because of the BIW sum rule, one needs only discuss ρ^2, from which the parallel comments on the inelastic contributions can be deduced trivially.

3.1 Non relativistic expansion

It is useful to comment briefly on the order of magnitude of the contributions in a non-relativistic expansion, i.e. in powers of v/c where v/c is now the *internal* velocity of the *light* quarks. We know that this velocity is not actually small ; nevertheless, this expansion gives more physical insight. The dominant contribution to ρ^2 is the first one ρ^2_{space}, from the spatial wave function. One finds at lowest order :

$$\rho^2\approx\rho^2_{space}\approx m^2(0|z^2|0)=\mathcal{O}\left((v^2/c^2)^{-1}\right) \tag{41}$$

$=\frac{m^2R^2}{2}$ in the h.o. case. This corresponds for τ to the use of the usual dipole formula. The Wigner rotation contribution is on the contrary highly suppressed :

$$\rho^2_{wigner}\approx\frac{1}{24}\left(0\left|\frac{\vec{p}^{\,2}}{m^2}\right|0\right)=\mathcal{O}(v^2/c^2) \quad . \tag{42}$$

$\rho^2_{dirac}=1/4$ is in between : $\mathcal{O}\left((v^2/c^2)^0\right)$; then it is actually the dominant relativistic effect coming from spin. This contrasts with Ref. [9] ; there, the discussion identifies the spin effect with the Wigner rotation effect only. Denoting ground state averages as $(\ )_0$, we can also write a v/c expansion of the full ρ^2 :

$$\rho^2 = m^2(z^2)_0 + \frac{1}{2}\left(\vec{p}^{\,2}z^2 + z^2\vec{p}^{\,2}\right)_0 + \frac{3}{4} + \mathcal{O}(v^4/c^4) \tag{43}$$

where the second term is $\mathcal{O}\left((v^2/c^2)^0\right)$ and comes for the "spatial" contribution ρ^2_{space}. It is seen that the lowest order relativistic corrections cannot be reduced in general, to the famous 1/4, as seems often assumed. Moreover they do not come from spin only ;

$$\rho^2_{space} = m^2\left(z^2\right)_0 + 1/2\left(\vec{p}^{\,2}z^2 + z^2\vec{p}^{\,2}\right)_0 + \frac{1}{4} + \mathcal{O}(v^4/c^4) \tag{44}$$

contributes to them as observed in Ref. [9]. Finally these additional relativistic corrections are not independent of the potential. Indeed $(\vec{p}^{\,2}z^2 + z^2\vec{p}^{\,2})_0$ has not even a definite sign.

Truly, it is a convention to take $m^2(z^2)_0$ as the starting point of the v/c expansion. Indeed, it must not be forgotten that the average are taken on internal wave functions, which in realistic cases shall include by themselves *relativistic binding corrections*. Consequently, since eq.(43) does not make explicit the latter relativistic effects, it is not the full v/c expansion around a truly nonrelativistic limiting case. Nevertheless, the relativistic corrections written in eq. (43) may be considered as a reasonable answer to the question inasmuch as we are concerned only with relativistic corrections to ρ^2 due to the center-of-mass motion of hadrons.

3.2 The lower bounds

It is obvious that in this model there is no upper bound since ρ^2 can be arbitrarily large in the nonrelativistic regime, e.g. $\frac{m^2R^2}{2}$ for an harmonic oscillator. On the other hand, we have found in the preceding paper[1] a lower bound Inf $\rho^2 = 3/4$, which is larger than 1/4 , the famous Bjorken lower bound. To investigate further the compositeness contribution $\rho^2_{space} + \rho^2_{wigner}$, it reveals useful to discuss the meaning of this bound. One may wonder why one cannot reach 1/4. Indeed, it would be tempting to suppose, by reverting the above argument, that one could reduce arbitrarily the "compositeness" contribution by reducing $m^2(z^2)_0$, which corresponds to going to a highly relativistic situation ($R^2 \to 0$). But the idea is wrong, due to the relativistic effects. Part of the reason is the Wigner rotation contribution ρ^2_{wigner}, which, for $|\vec{p}| \gg m$, amounts to 1/6. But another reason lies in the corrections to the "spatial" contribution ρ^2_{space}.

Indeed, when $m^2(z^2)_0$ is reduced by going to large average momenta, p^0 is increasing in average and then, the final outcome is that the full expression of ρ^2_{space}, $\left(0\left|\left(\frac{p^0z+zp^0}{2}\right)^2\right|0\right)$, is bounded from below. We can see this by the change of variable introduced in [1], such as $dx = p^0\frac{d}{dp}$:

$$p/m = \mathrm{sh}\ x\ ,\ p^0/m = \mathrm{ch}\ x \tag{45}$$

$$x = \mathrm{Argsh}\ p/m = \mathrm{Argch}\ p^0/m \tag{46}$$

x coincides nonrelativistically with p, but behaves as $\log p/m$ for large $p = |\vec{p}|$. We can write ρ^2_{space} as :

$$\left|\left|\frac{p^0z + zp^0}{2}\varphi_0\right|\right|^2 \tag{47}$$

where φ_0 is the space part of the 0^- wave function. Then, using rotation invariance of φ_0

$$\left\|\frac{p^0 z + z p^0}{2}\varphi_0\right\|^2 = \frac{1}{3}\left\|\left(p_0\frac{d}{dp}+\frac{1}{2}\frac{p}{p_0}\right)\varphi_0\right\|^2 = \frac{1}{3}\int d^3p\left|\left(\frac{d}{dx}+\frac{1}{2}\text{th x}\right)\varphi_0\right|^2 \quad . \tag{48}$$

Since $d^3p = 4\pi\ \text{sh}^2\text{x ch x dx}$ on spherically symmetric wave functions, we define :

$$g = \sqrt{4\pi}\ \text{sh x}\sqrt{\text{ch x}}\varphi_0(\text{m sh x}) \tag{49}$$

one then finds :

$$\left\|\left(p_0\frac{d}{dp}+\frac{1}{2}\frac{p}{p_0}\right)\varphi_0\right\|^2_{\vec{p}} = \left\|\left(\frac{d}{dx}-\text{th x}\right)g\right\|^2_x = \int dx\ g\left(1-\frac{d^2}{dx^2}\right)g = 1+\left\|\frac{dg}{dx}\right\|^2_x \quad . \tag{50}$$

The second term is positive and has 0 as lower bound, (obtained by spreading indefinitely $g(x)$). Therefore

$$\text{Inf}\left(0\left|\left(\frac{\text{p}_0\text{z}+\text{zp}_0}{2}\right)^2\right|0\right) = \frac{1}{3} \quad . \tag{51}$$

Note that φ_0 can be chosen arbitrary because the mass operator is arbitrary, except that $M_0 > 0$. Since the Wigner rotation contribution is positive and smaller than $1/6$ (since $p/p_0 + m \leq 1$), one gets : $7/12 \leq \text{Inf}\rho^2 \leq 9/12 = 3/4$. The more complete argument of [1] is necessary to show that :

$$\text{Inf}\ \rho^2 = \frac{3}{4} \tag{52}$$

exactly, i.e. the Wigner rotation attains its maximum at the lower bound of ρ^2. This reflects the fact that the bound is attained at large average of $|\vec{p}|$.

Since ρ^2_{space} is seen to be a major contribution to the lower bound $3/4 = 1/3 + 1/6 + 1/4$, it is worth specifying its origin. First it is an effect of compositeness, along with Wigner rotations. More specifically, it reflects the pure effect of *spatial* extension of the bound state. Finally, still more specifically, the bound $1/3$ in eq. (51) reflects the effect of the *relativistic* transformation on the spatial wave functions. Indeed, nonrelativistically ρ^2_{space} $(m^2(z^2)_0)$ could be made arbitrarily small . The effect of the relativistic transformation law amounts practically to replacing m by the larger p^0 (as noticed in [9]). Then the contribution is bounded from below as just demonstrated, and in accord with the qualitative guess $p^0 \sim |\vec{p}|$ for large $|\vec{p}|$ and $|\vec{p}||z| \sim 1$.

4 Conclusion

In conclusion, we see that the heavy quark limit of quark models for currents based on the Bakamjian-Thomas construction of states exactly satisfies the important sum rule or duality relation discovered by Bjorken and further analyzed by Isgur and Wise. This property is essentially due to the nontrivial fact that the wave functions in *motion* satisfy a closure relation. This, added to the previous demonstration that it is covariant and scales as required by the Isgur-Wise heavy quark symmetry relations, increases the interest of the

model. Another aspect illustrated by the present analysis is the capacity of quark models to give a physical insight in the saturation of general field theoretic relations through the use of the concepts of bound state physics. Finally, one is stimulated to investigate the subdominant regime in $1/m_Q$ which is involved in other important sum rules such as the Voloshin "optical" sum rule ; in general, properties such as covariance or the conservation of the current are lost, but could perhaps be recovered under certain conditions.

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